The Untwisted Stabilizer in Simple Current Extensions

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Abstract A method is presented to compute the order of the untwisted stabilizer of a simple current orbit, as well as some results about the properties of the resolved fields in a simple current extension.

In a recent paper [1], Fuchs, Schellekens and Schweigert presented an Ansatz to describe the modular properties of a CFT obtained by simple current extensions (for a review see [2]). An important role is played in their Ansatz by the so-called *untwisted stabilizer*, a subgroup of the ordinary stabilizer defined by some cohomological properties. In general it is quite difficult to determine the untwisted stabilizer, as it is related to the tranformation properties of the genus one holomorphic one-point blocks of the simple currents under the mapping class group action, and the latter data are not readily available.

The purpose of this note is to present a procedure that allows under suitable circumstances the determination of the order of the untwisted stabilizer solely from the knowledge of the $SL(2,\mathbf{Z})$ representation on the space of genus one characters of the original theory. The idea is to exploit the Frobenius-Schur indicator introduced for CFTs in [6].

We shall not go into the details of the FSS Ansatz [1], let's just recall the basic setting. We are given some CFT and a group $\mathcal{G}$ of integral spin simple currents, and we would like to construct the $SL(2,\mathbf{Z})$ representation of the new CFT obtained by extending the original one with the simple currents in $\mathcal{G}$. The first thing to do is of course to determine the primaries of the extended theory. The scheme is as follows :

1. First we keep only those primaries p of the original theory which have zero monodromy charge with respect to all the simple currents in $\mathcal{G}$, resulting in a set $I_0^{\mathcal{G}}$.

2. In the next step we identify those primaries from $I_0^{\mathcal{G}}$ that lie on the same $\mathcal{G}$-orbit, i.e. for which there exists a simple current in $\mathcal{G}$ transforming one into the other.

3. We split the so obtained orbits into several new primaries. Naively one would think that each orbit $[p]$ should be split into $|\,\mathcal{S}_p\,|$ new ones, where $\mathcal{S}_p = \{\alpha \in \mathcal{G} \,|\, \alpha p = p\}$ is the stabilizer of the orbit $[p]$, but it turns out that this so-called fixed point resolution [1] is governed by a subgroup $\mathcal{U}_p$ of the full stabilizer $\mathcal{S}_p$, the so-called *untwisted stabilizer* [1]. The actual definition

[1] As this last step is the crucial part of the construction, sometimes one refers to the extended theory as the *resolved* one.

of $\mathcal{U}_p$ involves the consideration of the space of genus one holomorphic one-point blocks for the simple currents in $\mathcal{G}$, thus it is in general a difficult problem to determine $\mathcal{U}_p$.

After having performed the above three steps, we are led to the following description of the primary fields of the extended theory : they are in one-to-one correspondence with pairs (p, ψ), where p is a $\mathcal{G}$-orbit of $I_0^{\mathcal{G}}$, and ψ is a linear character of $\mathcal{U}_p$, i.e. an element of the dual group $\hat{\mathcal{U}}_p$. As it cannot lead to confusion, we shall also denote by p any representative of the orbit p.

Now that we know how to describe the primaries, we can formulate the FSS Ansatz describing the $SL(2, \mathbf{Z})$ representation of the resolved theory. For the exponentiated conformal weights $\omega_p = \exp(2\pi\imath\Delta_p)$ of the primaries - i.e. the eigenvalues of the T-matrix - we have

$$\omega_{(p,\psi)} = \omega_p, \tag{1}$$

while for the S-matrix the FSS Ansatz reads [1]

$$S_{(p,\psi),(q,\chi)} = e_p e_q \frac{|\,\mathcal{G}\,|}{|\,\mathcal{S}_p\,||\mathcal{S}_q\,|} \sum_{\alpha \in \mathcal{U}_p \cap \mathcal{U}_q} \psi(\alpha) S_{pq}(\alpha) \bar{\chi}(\alpha), \tag{2}$$

where

$$e_p = [\mathcal{S}_p : \mathcal{U}_p]^{\frac{1}{2}} \tag{3}$$

is the square root of the index of $\mathcal{U}_p$ in $\mathcal{S}_p$, which is known to be an integer on general grounds, and $S_{pq}(\alpha)$ is the matrix element of the mapping class S acting on the space of genus one holomorphic one-point blocks of the simple current α, in the canonical basis of [3]. In particular, $S_{pq}(0) := S_{pq}$ is just the ordinary S-matrix of the original theory we started with. It may be shown [1, 3] that the above Ansatz leads to a consitent $SL(2, \mathbf{Z})$ representation for the extended theory, e.g. it satisfies the relations of [4].

Our aim is to determine the order of $\mathcal{U}_p$, or what is the same, the e_p-s from Eq. (3). The first thing to note is that, while to determine the actual matrix elements of S through Eq. (2) we need to know all of the matrix elements $S_{pq}(\alpha)$ - which are in general hard to compute -, upon summing over $\chi \in \mathcal{U}_q$ in Eq. (2), a standard argument of character theory yields

$$\sum_{\chi\in\hat{\mathcal{U}}_q} S_{(p,\psi),(q,\chi)} = \frac{e_p}{e_q}[\mathcal{G}:\mathcal{S}_p]S_{pq}. \tag{4}$$

This is the basic relation that we shall exploit in the sequel.

As a first corrolary of Eq. (4), by substituting for q the vacuum 0 in the above formula, we get that

$$S_{0(p,\psi)} = e_p[\mathcal{G}:\mathcal{S}_p]S_{0p}, \tag{5}$$

because no simple current fixes the vacuum, consequently $|\,\mathcal{S}_0\,| = e_0 = 1$. From this we get the following expression for the quantum dimensions of the fields

$$d_{(p,\psi)} = \frac{e_p d_p}{|\,\mathcal{S}_p\,|}. \tag{6}$$

Substituting Eq. (2) into Verlinde's formula [5]

$$N_{pq}^{r} = \sum_s \frac{S_{ps}S_{qs}\bar{S}_{rs}}{S_{0s}} \tag{7}$$

and exploiting Eq. (4) leads to

$$\sum_{\psi\in\hat{\mathcal{U}}_p,\chi\in\hat{\mathcal{U}}_q} N_{(p,\psi),(q,\chi)}^{(r,\rho)} = \frac{e_r}{e_p e_q}\frac{1}{|\,\mathcal{S}_r\,|}\sum_{\alpha\in\mathcal{G}} N_{pq}^{\alpha r}. \tag{8}$$

We are practically done, all that is left is to sustitute Eqs. (1), (5) and (8) into the formula for the Frobenius-Schur indicator [6]

$$\nu_p = \sum_{q,r} N_{qr}^{p} S_{0q}S_{0r}\frac{\omega_q^2}{\omega_r^2}, \tag{9}$$

to arrive at

$$\nu_{(p,\psi)} = \frac{e_p}{|\,\mathcal{S}_p\,|}\sum_{\alpha\in\mathcal{G}} \mathcal{Z}[p,\alpha], \tag{10}$$

where $\mathcal{Z}[p,\alpha]$ stands for the $\mathcal{Z}$-matrix introduced in [6], with matrix elements

$$\mathcal{Z}[p,q] = \omega_q^{-\frac{1}{2}}\sum_{r,s} N_{rs}^{p} S_{qr}S_{0s}\frac{\omega_s^2}{\omega_r^2}. \tag{11}$$

If we introduce the notation

$$\mathcal{Z}_{\mathcal{G}}(p) := \frac{1}{|\,\mathcal{S}_p\,|}\sum_{\alpha\in\mathcal{G}} \mathcal{Z}[p,\alpha], \tag{12}$$

then because the lhs. of Eq. (10) can only take the values ± 1 and 0, we get the following alternative :

1. $\mathcal{Z}_{\mathcal{G}}(p) = 0$. In this case the resolved field (p, ψ) is complex, and we get no information on e_p from Eq. (10).

2. $\mathcal{Z}_{\mathcal{G}}(p) \neq 0$. In this case the resolved field (p, ψ) is either real or pseudo-real according to the sign of $\mathcal{Z}_{\mathcal{G}}(p)$, moreover

$$e_p = \frac{1}{\mid \mathcal{Z}_{\mathcal{G}}(p) \mid}, \tag{13}$$

or in other words

$$\mid \mathcal{U}_p \mid \; = \; \mid \mathcal{S}_p \mid \mathcal{Z}_{\mathcal{G}}^2(p). \tag{14}$$

We see that at least for some of the primaries p we can determine e_p from the knowledge of $\mathcal{Z}_{\mathcal{G}}(p)$, which is in turn completely determined by the $SL(2, \mathbf{Z})$ action on the space of genus one characters of the original theory.

What to do for those primaries for which $\mathcal{Z}_{\mathcal{G}}(p) = 0$? Well, in that case there is still some hope to determine e_p through a similar procedure, because Eq. (10) is just a special case of the more general relation

$$\sum_{\chi \in \hat{\mathcal{U}}_q} \mathcal{Z}\left[(p, \psi), (q, \chi)\right] = \frac{e_p}{e_q} \frac{1}{\mid \mathcal{S}_p \mid} \sum_{\alpha \in \mathcal{G}} \mathcal{Z}[p, \alpha q]. \tag{15}$$

Exploiting the properties of the $\mathcal{Z}$-matrix [6], namely that its matrix elements $\mathcal{Z}[p, q]$ are integers of the same parity as N_{pp}^{q} and bounded in absolute value by the latter, sometimes it is possible to get from Eq. (15) the index e_p for primaries for which Eq. (10) does not yield an answer.

In summary, we have seen that a great deal of information about the extended theory may be learned simply by looking at the $SL(2, \mathbf{Z})$ action on the space of characters of the original theory, e.g. in some cases the indices e_p may be determined without having to look at the mapping class group action on the space of genus one holomorphic one-point blocks. While the knowledge of the e_p-s is not enough in general to determine $\mathcal{U}_p$ itself, it is nevertheless enough for the correct counting of the primary fields of the resolved theory and the computation of their quantum dimensions and other characteristics. Moreover, in many practical instances e_p completely determines the untwisted stabilizer, e.g. in the obvious cases $e_p = 1$ and $e_p^2 = \mid \mathcal{S}_p \mid$.

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